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Classification of three-distance sets in two dimensional Euclidean space

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Abstract

A subset X in k-dimensional Euclidean space $\mathbb{R}^k$ is called an s-distance set if there are exactly s distances between two distinct points in X. S.J. Einhorn–I.J. Schoenberg conjectured that there are only five maximal (i.e. cannot be contained in others) three-distance sets in $\mathbb{R}^2$ having five or more points. In this paper, we show that there are in fact twenty four maximal three-distance sets in $\mathbb{R}^2$ having five or more points and determine the largest possible cardinality of three-distance sets in $\mathbb{R}^2$.

Keywords: Three-distance sets; Euclidean space

1. Introduction

Let $\mathbb{R}^k$ be a k-dimensional Euclidean space. For $X \subset \mathbb{R}^k$, let $A(X) = \{d(x, y) \mid x, y \in X, x \neq y\}$ where $d(x, y)$ is the Euclidean distance between x and y in $\mathbb{R}^k$. We call X an s-distance set if $|A(X)| = s$. For two subsets in $\mathbb{R}^k$, we say that they are isomorphic if there exists similar transformation from one to the other. An interesting problem on s-distance sets is to determine the largest possible cardinality of s-distance sets in $\mathbb{R}^k$. We denote this number by $n(k, s)$. Bannai et al. [2] and Blokhuis [3] gave an upper bound $n(k, s) \leq \binom{k+s}{s}$. For $s = 2$, the numbers $n(k, 2)$ are known for $k \leq 8$ (Kelly [7], Croft [4], and Lisoněk [9]). However, for $s \geq 3$, even the smallest case $n(2, 3)$ had not been determined. On three-distance sets, Einhorn–Schoenberg [6] conjectured the following:

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(i) The only maximal three-distance sets in $\mathbb{R}^2$ having five or more points, are the vertices of the regular k-gon together with its center for $k = 4, 5, 6$, seven points of the regular heptagon and six points of the fig. 603.

(ii) The points of the regular icosahedron form the only three-distance set in $\mathbb{R}^3$ having twelve points.

The next theorem is the main result of this paper.

Theorem 1. *There are thirty four three-distance sets having five points in $\mathbb{R}^2$ to within isomorphism.*

By Theorem 1, we classify three-distance sets having more than five points and show that $n(2, 3) = 7$.

Theorem 2. (i) *There is no three-distance set having more than seven points in $\mathbb{R}^2$.*

(ii) *There exist only two maximal three-distance sets having seven points in $\mathbb{R}^2$.*

(iii) *There exist only six maximal three-distance sets having six points in $\mathbb{R}^2$.*

(iv) *There are only sixteen maximal three-distance sets having five points in $\mathbb{R}^2$.*

In particular, there are exactly twenty four maximal three-distance sets having five or more points in $\mathbb{R}^2$.

Throughout this paper, we deal with distance sets in $\mathbb{R}^2$ unless we note otherwise.

2. Graph representation

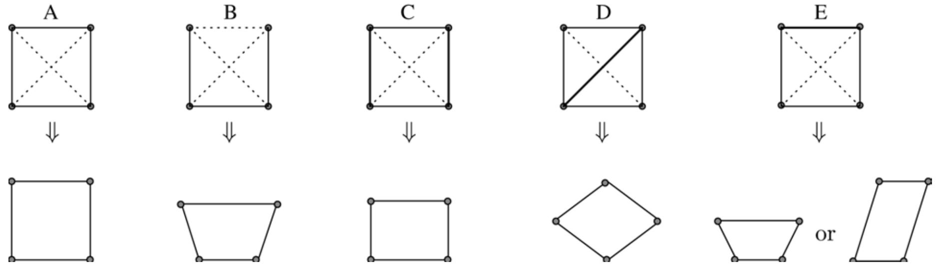
Einhorn–Schoenberg [5] classified two-distance sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ using a correspondence of a two-distance set to a simple graph. In this paper we generalize this correspondence. First we prepare some definitions for a graph. Let $G = (V, E)$ be a simple graph where $V = V(G)$ and $E = E(G)$ are the vertex set and the edge set of G respectively. If two vertices $u, v \in V(G)$ are connected, we denote this by uv. For any subset U of $V(G)$, we let $\mathrm{Ind}(U) = \mathrm{Ind}(G, U)$ be the subgraph of G induced by U, that is, two vertices $u, v \in U$ are connected if they are in G. We call $\mathrm{Ind}(U)$ the induced subgraph of G on U. Let $B = \{b_1, b_2, \ldots, b_s\}$ be a set of cardinality s. An s colored graph of order n by B is a pair $h(K_n) = (K_n, h)$ of a complete graph K_n of order n and a surjection $h : E(K_n) \to B$ (called an s coloring by B).

Two s colored graphs $h(K_n)$ by B and $h'(K_n)$ by B' are said to be isomorphic if there exists a bijection $\iota : B \to B'$ such that $\iota \circ h = h'$. We define induced subgraphs of an s colored graph in natural manner.

Definition (Graph Representation for s-Distance Set). Let $X \subset \mathbb{R}^k$ be an s-distance set and $h(K_n)$ an s colored graph by B. A graph representation of X to $h(K_n)$ is a pair of bijections $f : X \to V(K_n)$, $g : A(X) \to B$ such that $h(f(p_1) f(p_2)) = g(d(p_1, p_2))$ for any $p_1, p_2 \in X$.

A graph representation of an s-distance set assigns an s colored graph. Conversely when the s colored graph $h(K_n)$ is given, if there exist an s-distance set X in $\mathbb{R}^k$ and a graph representation of X to $h(K_n)$, then X is called an embedding of $h(K_n)$ in $\mathbb{R}^k$. The following are examples for embeddings in $\mathbb{R}^2$ which will play an important role in Section 3.2.2.

Example. The heavy lines, thin lines and dotted lines mean the colors of the edges.



The colored graph A is embedded as four points of a square, the graph B is embedded as four points of regular pentagon, and the graph C is embedded as four points of a rectangle. The colored graph D is embedded as four points of a rhombus, the graph E is embedded as four points of an isosceles trapezoid whose diagonal lines correspond to the thin lines or dotted lines, or four points of a parallelogram whose sides correspond to the thin lines and dotted lines.

Note that the embeddings of A and B are uniquely determined to within the isomorphism, but there are infinitely many embeddings for C, D and E.

3. Classification of three-distance sets having five points

To classify three-distance sets having five points, we proceed as follows:

(1) Those containing two-distance sets having four points.

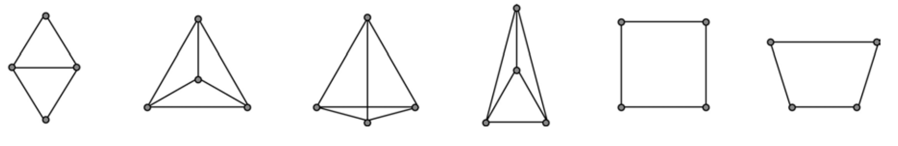
(2) Those not containing two-distance sets having four points.

 (2.1) Those containing the three points of an equilateral triangle.

 (2.2) Those not containing the three points of an equilateral triangle.

3.1. Containing two-distance sets having four points

We consider three-distance sets which contain two-distance sets having four points. Einhorn–Schoenberg [6] completely classified the two-distance sets having four points as follows.



In this case, we prove the following lemma.

Lemma 3. *There exist only finitely many three-distance sets having five points and containing two-distance sets having four points.*

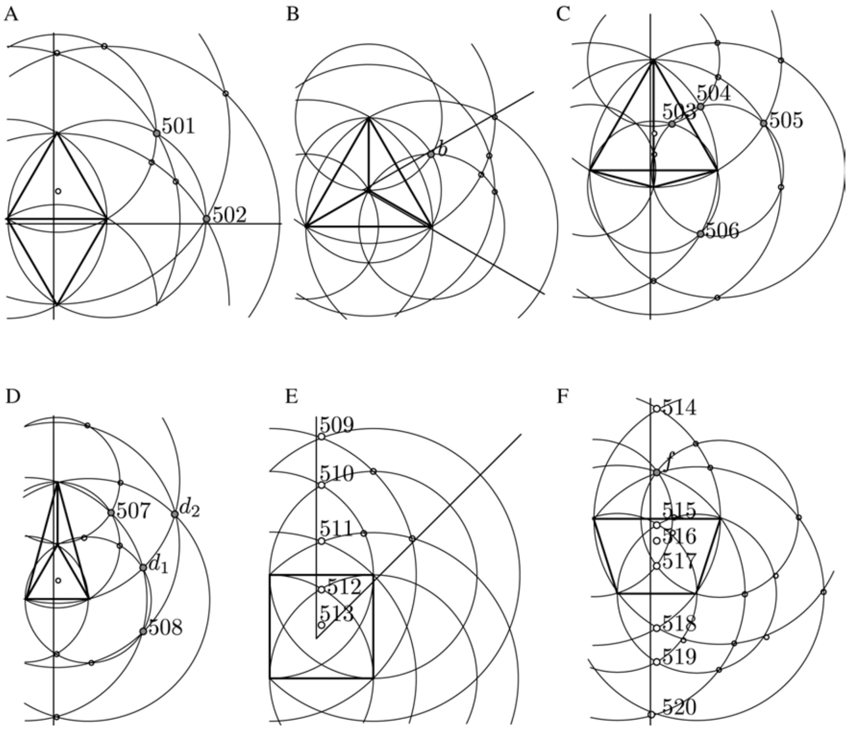


Fig. 1.

Proof. Let Y be one of the two-distance sets having four points with $A(Y) = \{\alpha, \beta\}$. For a new point $p \in \mathbb{R}^2$, let $O_p(Y) = \{q \in Y \mid d(p, q) \in A(Y)\}$. Then $X = Y \cup \{p\}$ is a three-distance set with $A(X) = \{\alpha, \beta, \gamma\}$ if and only if $Y \neq O_p(Y)$ and $d(p, q) = \gamma$ for any $q \in Y \backslash O_p(Y)$. Note that $|O_p(Y)|$ is the number of cycles which passes the point p whose center is one of the elements of Y and whose radius is either α or β. Since there are only finitely many points of intersection of these cycles, if $|O_p(Y)| \geq 2$, then the number of these three-distance sets is bounded. If $|O_p(Y)| \leq 1$, then $|Y \backslash O_p(Y)| \geq 3$. If $Y \cup \{p\}$ is a three-distance set then $d(p, q) = \gamma$ for any $q \in Y \backslash O_p(Y)$. This means that p is the center of the cycle which passes each element of $Y \backslash O_p(Y)$. Since there are only finitely many possibilities for such cycles, the number of these three-distance sets is finite. $\square$

By Lemma 3, we can classify easily such three-distance sets. In Fig. 1, for each two-distance set above, we list possible candidates which can be the fifth points of three-distance sets. Among these points, the labeled points are the ones which actually form three-distance sets. Here we consider the symmetry of the two-distance sets.

For each labeled point p, we let stand for fig. p the three-distance set obtained by adding p. The three-distance set fig. b is isomorphic to fig. 502, fig. d_1 is isomorphic to fig. 503, fig. d_2 is isomorphic to fig. 506. The point f is an exception since fig. f is a two-distance set.

3.2. *Not containing two-distance sets having four points*

3.2.1. *Containing three points of equilateral triangles*

In this case, we may put $X = T \cup \{p, q\}$ where $T = \{t_1, t_2, t_3\}$ is the set of three points of the equilateral triangle. We assume that $d(t_i, t_j) = 1$ for $1 \le i < j \le 3$ without loss of generality. Let $A(X) = \{1, \alpha, \beta\}$ $(\alpha < \beta)$. Let $\mathbb{M}$ be the union of cycles with radius 1 and centers in T, and $\mathbb{L}$ the union of perpendicular bisectors of the sides of the equilateral triangle. Then we have the following lemma.

Lemma 4. *Let $X = T \cup \{p, q\}$ be a three-distance set having five points not containing two-distance sets having four points. Then p and q satisfy one of the following:*

(a) $p, q \in \mathbb{M} \backslash \mathbb{L}$.
(b) $p, q \in \mathbb{L} \backslash \mathbb{M}$.
(c) $p \in \mathbb{M}$ *and* $q \in \mathbb{L}$ *(or* $q \in \mathbb{M}$ *and* $p \in \mathbb{L}$*).*

Proof. We consider the graph representation corresponding to the three-distance set. Let V be the vertex set corresponding to the three-distance set, $V_3 = \{v_1, v_2, v_3\}$ the set corresponding to the three points of T, and the edge between two points of V_3 be colored by b, that is, $h(vw) = b$ for $v, w \in V_3$. Let

$$M = \bigcup_{i=1}^{3} M_i, \qquad \text{where } M_i = \{v \in V \backslash V_3 \mid h(vv_i) = b\},$$

$$L = \bigcup_{1 \le i < j \le 3} L_{ij}, \qquad \text{where } L_{ij} = \{v \in V \backslash V_3 \mid h(vv_i) = h(vv_j)\}.$$

The vertices in M correspond to the points on $\mathbb{M}$ and the vertices in L correspond to the points on $\mathbb{L}$. For any vertex $w \in V \backslash V_3$, if $w \in M \cap L$, then $V \cup \{w\}$ corresponds to a two-distance set, which contradicts the assumption. Hence $w \notin M \cap L$. If $w \notin M$, $v_i w$ for $i = 1, 2, 3$ are colored by the remaining two colors. This means that one of the colors is used at least twice. Therefore if $w \notin M$, we have $w \in L$. $\square$

(a) Here we consider the three-distance set $X = \{t_1, t_2, t_3, p, p'\}$ with $p, p' \in \mathbb{M} \backslash \mathbb{L}$, containing no two-distance set having four points. Since X does not contain two-distance sets having four points, if $p \in \mathbb{M} \backslash \mathbb{L}$, then α and β are uniquely determined by p. We take five points which have such α and β except p. We denote the five points p_i $(i = 1, 2, \ldots, 5)$.

In Fig. 2, the points on the heavy line are the points which are on a cycle and outside of the other two cycles, so that the distances satisfy $1 < \alpha < \beta$. Similarly the points on the thin line satisfy $\alpha < 1 < \beta$ and the points on the dotted line satisfy $\alpha < \beta < 1$. So we divide in the following cases: (i) $1 < \alpha < \beta$, (ii) $\alpha < 1 < \beta$, (iii) $\alpha < \beta < 1$. Using the symmetry, it is enough to check the distance $d(p, p_i)$ and if $X = \{t_1, t_2, t_3, p, p_i\}$ can be a three-distance set for $i = 1, 2, 3, 4$.

(i) The case where $1 < \alpha < \beta$.

We take points $p, p_1, \ldots, p_4, t_1, t_2, t_3$ as in Fig. 3. We check if the set $X = \{t_1, t_2, t_3, p, p_i\}$ is a three-distance set belonging to the class (a). Now we have $\alpha = d(p, t_2)$, $\beta = d(p, t_3)$.

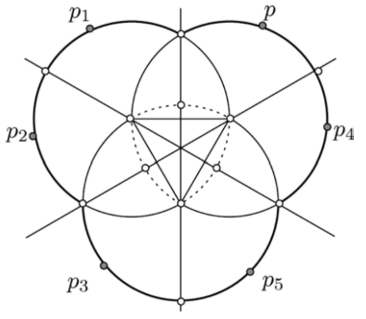


Fig. 2.

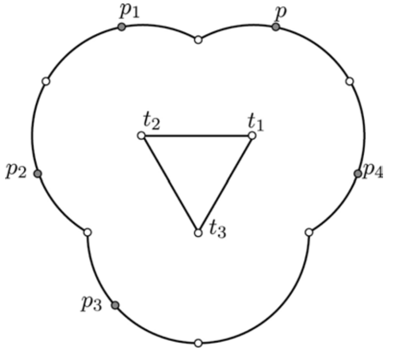


Fig. 3.

1. Let $X = \{t_1, t_2, t_3, p, p_1\}$ and $d = d(p, p_1)$.
   If $d = 1$, $\{p, p_1, t_1, t_2\}$ is a two-distance set.
   If $d = \alpha$, $\{p, p_1, t_1, t_2\}$ is a two-distance set.
   If $d = \beta$, then X is a desired three-distance set such that $\angle pt_3 p_1 = \frac{\pi}{3}$ (fig. 521).
2. Let $X = \{t_1, t_2, t_3, p, p_2\}$ and $d = d(p, p_2)$.
   Since $\angle pt_3 p_2 = \frac{\pi}{2}$, we have $d > d(p, t_3) = \beta$.
3. Let $X = \{t_1, t_2, t_3, p, p_3\}$ and $d = d(p, p_3)$.
   Since $\angle pt_3 p_3 > \frac{\pi}{2}$, we have $d > d(p, t_3) = \beta$.
4. Let $X = \{t_1, t_2, t_3, p, p_4\}$ and $d = d(p, p_4)$.
   If $d = 1$, then X is a three-distance set in this class such that $\angle pt_1 p_4 = \frac{\pi}{3}$ (fig. 522).
   If $d = \alpha$, since $d(t_2, p) = d(p, p_4) = d(p_4, t_3)$, X is a three-distance set such that $\angle pt_1t_2 = \angle pt_1 p_4 = \angle p_4t_1t_3 = \frac{5\pi}{9}$ (fig. 523).
   If $d = \beta$, we have $\angle pt_1t_2 = \angle p_4t_1t_3$ and $\angle pt_1t_3 = \angle pt_1 p_4$ so that X is also a three-distance set such that $\angle pt_1 p_4 = \frac{7\pi}{9}$ (fig. 524).

(ii) The case where $\alpha < 1 < \beta$.

We take points $p, p_1, \ldots, p_4, t_1, t_2, t_3$ as in Fig. 4. We check if the set $X = \{t_1, t_2, t_3, p, p_i\}$ is a three-distance set belonging to the class (a). Now we have $\alpha = d(p, t_1)$, $\beta = d(p, t_3)$.

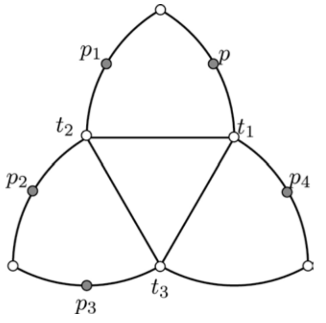


Fig. 4.

1. Let $X = \{t_1, t_2, t_3, p, p_1\}$ and $d = d(p, p_1)$ (<1).
   If $d = \alpha$, $\{p, p_1, t_1, t_2\}$ is a two-distance set.
2. Let $X = \{t_1, t_2, t_3, p, p_2\}$ and $d = d(p, p_2)$.
   Since $d(p, t_2) = d(t_3, t_1)$, $d(t_2, p_2) = d(t_1, p)$ and $\angle pt_2p_2 > \angle t_3t_1p$, we have $d > d(p, t_3) = \beta$.
3. Let $X = \{t_1, t_2, t_3, p, p_3\}$ and $d = d(p, p_3)$.
   Since $\angle pt_3p_3 > \frac{\pi}{2}$, we have $d > d(p, t_3) = \beta$.
4. Let $X = \{t_1, t_2, t_3, p, p_4\}$ and $d = d(p, p_4)$.
   Since $\angle pt_1p_4 > \frac{\pi}{3}$, we have $d > d(t_1, p) = \alpha$.
   If $d = 1$, $\{p, p_4, t_2, t_3\}$ is a two-distance set.
   If $d = \beta$, $\{p, p_4, t_2, t_3\}$ is again a two-distance set.

(iii) The case where $\alpha < \beta < 1$.

We take points $p, p_1, \ldots, p_4, t_1, t_2, t_3$ as in Fig. 5. We check if the set $X = \{t_1, t_2, t_3, p, p_i\}$ is a three-distance set belonging to the class (a). Now we have $\alpha = d(p, t_1)$, $\beta = d(p, t_2)$.

1. Let $X = \{t_1, t_2, t_3, p, p_1\}$ and $d = d(p, p_1)$ (<1).
   Since $\angle pp_1t_2 > \frac{\pi}{2}$, we have $d < d(p, t_2) = \beta$.
   If $d = \alpha$, then since $d(t_1, p) = d(p, p_1) = d(p_1, t_2)$, X is a three-distance set such that $\angle t_1t_3p = \angle pt_3p_1 = \angle p_1t_3t_2 = \frac{\pi}{9}$ (fig. 525).
2. Let $X = \{t_1, t_2, t_3, p, p_2\}$ and $d = d(p, p_2)$.
   Since $\angle p_2pt_1 > \frac{\pi}{2}$, we have $d < d(p_2, t_1) = 1$.
   Since $\angle pt_1t_2 < \angle pt_1p_2$, we have $d > d(p, t_2) = \beta$.
3. Let $X = \{t_1, t_2, t_3, p, p_3\}$ and $d = d(p, p_3)$ (<1).
   Since $\angle pt_2p_3 > \frac{\pi}{3}$, we have $d > d(p, t_2) = \beta$.
4. Let $X = \{t_1, t_2, t_3, p, p_4\}$ and $d = d(p, p_4)$ (<1).
   Since $\angle pt_1p_4 > \frac{\pi}{3}$, we have $d > d(p, t_1) = \alpha$.
   If $d = \beta$, $\{p, p_4, t_2, t_3\}$ is a two-distance set.

(b) Here we consider three-distance set $X = \{t_1, t_2, t_3, p, p'\}$ with $p, p' \in \mathbb{L}\backslash\mathbb{M}$, containing no two-distance sets having four points. Since X does not contain two-distance sets having four points, if $p \in \mathbb{L}\backslash\mathbb{M}$, then α and β are uniquely determined by p. We take

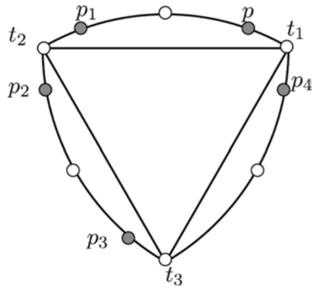


Fig. 5.

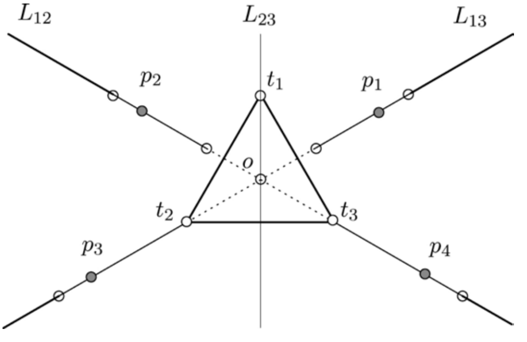


Fig. 6.

at most two points which have such α and β on each perpendicular bisector $L_{i,j}$ where $1 \le i < j \le 3$. For the symmetry, we may consider that there exist two points p, p' on the L_{12} and L_{13}. Similarly to part (a), we divide in the following cases: (i) $1 < \alpha < \beta$ (heavy lines), (ii) $\alpha < 1 < \beta$ (thin lines), (iii) $\alpha < \beta < 1$ (dotted lines). In any case, we take points $o, t_1, t_2, t_3, p_1, p_2, p_3, p_4$ as in Fig. 6. It is enough to check $d = d(p_i, p_j)$ and if $X = \{t_1, t_2, t_3, p_i, p_j\}$ can be a three-distance set for $(i, j) = (1, 2), (1, 3), (1, 4), (3, 4)$.

(i) The case where $1 < \alpha < \beta$.

1. Let $X = \{t_1, t_2, t_3, p_1, p_2\}$ and $d = d(p_1, p_2)$.
   Since $\angle p_1 t_2 p_2 > \frac{\pi}{2}$, we have $d > d(p_1, t_2) = \beta$.
2. Let $X = \{t_1, t_2, t_3, p_1, p_3\}$ and $d = d(p_1, p_3)$.
   Clearly $d > d(p_1, t_2) = \beta$.
3. Let $X = \{t_1, t_2, t_3, p_1, p_4\}$ and $d = d(p_1, p_4)$.
   Since $\angle p_1 t_2 p_4 < \angle p_1 o p_4 = \frac{\pi}{3}$, we have $d < d(p_1, t_2)(= d(p_4, t_2)) = \beta$.
   Since $\angle p_1 t_3 p_4 > \angle p_1 o p_4 = \frac{\pi}{3}$, we have $d > d(p_1, t_3)(= d(p_4, t_3)) = \alpha$.
   So we have $\alpha < d < \beta$.
4. Let $X = \{t_1, t_2, t_3, p_3, p_4\}$ and $d = d(p_3, p_4)$.
   Since $\angle p_3 t_1 p_4 > \frac{\pi}{2}$, we have $d > d(p_3, t_1)(= d(p_4, t_1)) = \beta$.

(ii) The case where $\alpha < 1 < \beta$.

1. Let $X = \{t_1, t_2, t_3, p_1, p_2\}$ and $d = d(p_1, p_2)$.
   Since $\angle p_1 t_1 p_2 > \frac{\pi}{3}$, we have $d > d(p_1, t_1)(= d(p_2, t_1)) = \alpha$.
   If $d = 1$, then X is a three-distance set such that $d(p_1, t_1) = d(p_1, t_3)$, $\angle p_2 p_1 t_3 = \frac{\pi}{2}$ (fig. 526).
   If $d = \beta$, then since $\angle p_1 t_2 t_3 = \angle p_2 p_1 t_2 = \frac{\pi}{6}$, X is a three-distance set such that $\angle p_2 p_1 t_3 = \frac{5\pi}{12}$, $\angle t_1 p_1 t_3 = \frac{\pi}{2}$ (fig. 527).
2. Let $X = \{t_1, t_2, t_3, p_1, p_3\}$ and $d = d(p_1, p_3)$.
   Clearly $d > d(p_1, t_2) = \beta$.
3. Let $X = \{t_1, t_2, t_3, p_1, p_4\}$ and $d = d(p_1, p_4)$.
   By an argument similar to that in (i), we have $\alpha < d < \beta$.
   If $d = 1$, then since $d(t_1, p_1) = d(p_1, t_3) = d(t_3, p_4)$, $d(t_1, t_3) = d(p_1, p_4)$, we have $t_1 t_3 // p_1 p_4$. Thus we have $\beta = d(p_1, t_2) < d(p_4, t_2) = \beta$ since $\angle t_2 p_1 p_4 = \frac{\pi}{2}$, which is impossible.
4. Let $X = \{t_1, t_2, t_3, p_3, p_4\}$ and $d = d(p_3, p_4)$.
   Since $\angle p_3 t_1 p_4 > \frac{\pi}{3}$, $d > d(p_3, t_1)(= d(p_4, t_1)) = \beta$.

(iii) The case where $\alpha < \beta < 1$.

1. Let $X = \{t_1, t_2, t_3, p_1, p_2\}$ and $d = d(p_1, p_2)$.
   Since $\angle p_1 p_2 t_2 > \frac{\pi}{2}$, $d < d(p_1, t_2) = \beta$.
   If $d = \alpha$, then since $\angle p_1 t_1 p_2 = \frac{\pi}{3}$, X is isomorphic to fig. 521.
2. Let $X = \{t_1, t_2, t_3, p_1, p_3\}$ and $d = d(p_1, p_3)$.
   Since $\alpha = d(p_3, t_2) = d(p_1, t_1) \geq \frac{1}{2}$, if $d \geq \frac{1}{2}$, then $\beta = d(p_1, t_2) = d + \alpha \geq 1$.
3. Let $X = \{t_1, t_2, t_3, p_1, p_4\}$ and $d = d(p_1, p_4)$.
   Since $\angle p_1 t_3 p_4 < \frac{\pi}{3}$, we have $d < d(p_1, t_3)(= d(p_4, t_3)) = \alpha$.
4. Let $X = \{t_1, t_2, t_3, p_3, p_4\}$ and $d = d(p_3, p_4)$.
   Since $\angle p_3 t_1 p_4 < \frac{\pi}{3}$, we have $d < d(p_3, t_1)(= d(p_4, t_1)) = \beta$.
   If $d = \alpha$, then $\angle p_3 t_2 t_3 = \frac{\pi}{6}$, $\angle t_2 p_3 p_4 = \frac{5\pi}{6}$, $\angle p_3 t_2 p_4 = \frac{\pi}{12}$, and hence $X = \{t_1, t_2, t_3, p_3, p_4\}$ is a three-distance set such that $\angle t_1 p_3 t_3 = \angle t_1 p_4 t_2 = \frac{\pi}{2}$ (fig. 028).

(c) Let $p \in \mathbb{L}$, $q \in \mathbb{M}$. We consider whether $\{t_1, t_2, t_3, p, q\}$ can be a three-distance set not containing two-distance sets having four points. It is necessary that there exist $p \in \mathbb{L}$, $q \in \mathbb{M}$ such that $A(X) = A(X')$ where $X = \{t_1, t_2, t_3, p\}$, $X' = \{t_1, t_2, t_3, q\}$. Let $A(X) = \{1, \alpha, \beta\}$ $(\alpha < \beta)$, $A(X') = \{1, \alpha', \beta'\}$ $(\alpha' < \beta')$. Here we divide in the following cases: (i) $1 < \alpha < \beta$ and $1 < \alpha' < \beta'$, (ii) $\alpha < 1 < \beta$ and $\alpha' < 1 < \beta'$, (iii) $\alpha < \beta < 1$ and $\alpha' < \beta' < 1$. In any case, we consider whether a triangle consisting of three sides of the length $1, \alpha, \beta$ is congruent with a triangle consisting of three sides of the length $1, \alpha', \beta'$.

(i) The case where $1 < \alpha < \beta$ and $1 < \alpha' < \beta'$.

We take the points as in Fig. 7. Note either $p = p_1$ or $p = p_2$. Then $(\alpha, \beta) = (d(p_1, t_1), d(p_1, t_3))$, $(d(p_2, t_3), d(p_2, t_1))$. By the symmetry, q moves between o and o' so that we have $(\alpha', \beta') = (d(q, t_2), d(q, t_3))$.

For the angle included by the sides of the length α' and β', we have

$$\angle t_2 q t_3 = \frac{\pi}{6}.$$

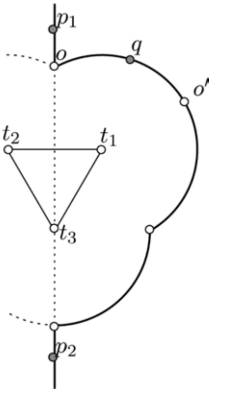


Fig. 7.

For the angle included by the sides of the length α and β, we have

$$\angle t_1 p_1 t_3,\ \angle t_1 p_2 t_3 < \frac{\pi}{6}.$$

Therefore $A(X) \neq A(X')$.

(ii) The case where $\alpha < 1 < \beta$ and $\alpha' < 1 < \beta'$.

We take the points as in Fig. 8. Note either $p = p_1$ or $p = p_2$. Then $(\alpha, \beta) = (d(p_1, t_1), d(p_1, t_3)),\ (d(p_2, t_3), d(p_2, t_1))$. By the symmetry, q moves between o and t_1 so that we have $(\alpha', \beta') = (d(q, t_1), d(q, t_3))$.

For the angle included by the sides of the length 1 and α', we have

$$\frac{2\pi}{3} < \angle q t_1 t_3 < \frac{5\pi}{6}.$$

For the angle included by the sides of the length 1 and α, we have

$$\angle p_1 t_1 t_3 < \frac{2\pi}{3},\ \angle p_2 t_3 t_1 = \frac{5\pi}{6}.$$

Therefore $A(X) \neq A(X')$.

(iii) The case where $\alpha < \beta < 1$ and $\alpha' < \beta' < 1$.

We take the points as in Fig. 9. Note either $p = p_1$ or $p = p_2$. Then $(\alpha, \beta) = (d(p_1, t_1), d(p_1, t_3)),\ (d(p_2, t_3), d(p_2, t_1))$. By the symmetry, q moves between t_1 and o. So that we have $(\alpha', \beta') = (d(q, t_1), d(q, t_3))$.

For the angle included by the sides of the length α' and 1,

$$\text{since } \angle t_1 q t_3 = \frac{5\pi}{6}, \qquad \text{we have } \angle q t_1 t_3 < \frac{\pi}{6}.$$

For the angle included by the sides of the length α and 1, we have

$$\frac{\pi}{6} < \angle p_1 t_1 t_3,\ \angle p_2 t_3 t_1 = \frac{\pi}{6}.$$

Therefore $A(X) \neq A(X')$.

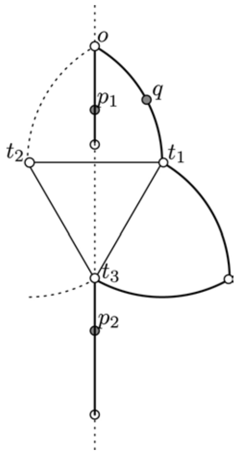


Fig. 8.

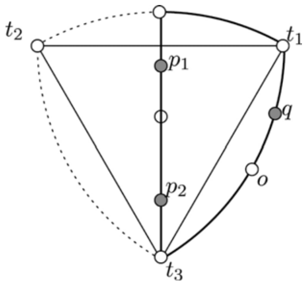


Fig. 9.

Thus we do not have such three-distance sets in this case.

3.2.2. Not containing three points of an equilateral triangle

In this case, we subdivide in the next two cases:

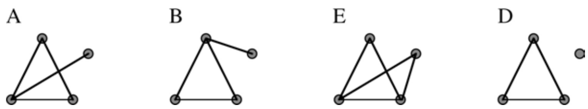
(a) For each $\gamma \in A(X)$, there exists an isosceles triangle whose two sides have length γ.
(b) There exists $\gamma \in A(X)$ such that there is no isosceles triangle whose two sides have length γ.

First of all, we consider case (a). In this case, in particular the three-distance sets contain an isosceles triangle whose two sides have the longest length. First we consider graph representation for the four-element subsets containing the isosceles triangle. We denote the edges which correspond to the longest sides by heavy lines, the edges which correspond to the sides with length equal to that of the base of the isosceles triangle by thin lines, and the other edges by dotted lines. Since these four-element subsets do not contain equilateral triangles, there are the following four possibilities for the remaining heavy lines.

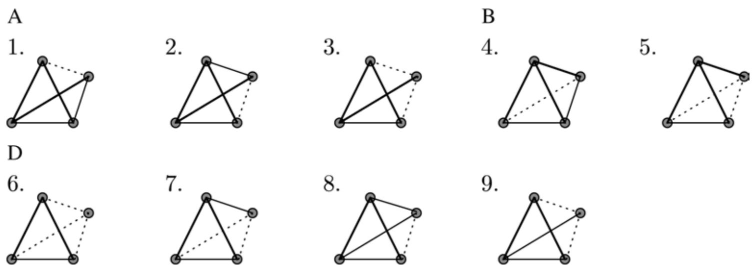
Table 1

i	θ	$p_i(\theta)$	$\beta_i(\theta)$
1	$\frac{\pi}{12} < \theta < \frac{\pi}{6}$	$(\cos A - \sin\theta, \sin A)$	$\sqrt{2 - 2\sin(A+\theta)}$
2	$0 < \theta < \frac{\pi}{6}$	$(\sin 3\theta - \sin\theta, \cos 3\theta)$	$\sqrt{1 - 4\sin 3\theta \sin\theta + 4\sin^2\theta}$
3	$0 < \theta < \frac{\pi}{6}$	$(\cos B - \sin\theta, \sin B)$	$\sqrt{1 - 4\sin\theta\cos B + 4\sin^2\theta}$
4	$0 < \theta < \frac{\pi}{12}$	$(\sin 3\theta, \cos\theta - \cos 3\theta)$	$2\sin 2\theta$
5	$0 < \theta < \frac{\pi}{6}$	$(0, \cos\theta - 1)$	$\sqrt{2 - 2\cos\theta}$
6	$0 < \theta < \frac{\pi}{6}$	$\left(0, \cos\theta - \frac{1}{2\cos\theta}\right)$	$\frac{1}{2\cos\theta}$
7	$0 < \theta < \frac{\pi}{6}$	$(0, \cos\theta - 2\sin\theta)$	$\sqrt{1 - 4\sin\theta\cos\theta + 4\sin^2\theta}$
8-1	$\arcsin\frac{1}{4} \le \theta < \frac{\pi}{6}$	$(\beta_{81}\cos C - \sin\theta, \beta_{81}\sin C)$	$4\sin\theta\cos C$
8-2	$\arcsin\frac{1}{4} < \theta < \frac{\pi}{10}$	$(\beta_{82}\cos C' - \sin\theta, \beta_{82}\sin C')$	$4\sin\theta\cos C'$
9-1	$\arcsin\left(\frac{\sqrt{2}-1}{2}\right) \le \theta < \frac{\pi}{12}$	$(m\cos\theta + c, m\sin\theta)$	$\sqrt{(b+a)^2 + \frac{1}{4}}$
9-2	$\arcsin\left(\frac{\sqrt{2}-1}{2}\right) < \theta < \frac{\pi}{6}$	$(m'\cos\theta + c, m'\sin\theta)$	$\sqrt{(b-a)^2 + \frac{1}{4}}$

$\theta \ne \frac{\pi}{10}$ for $i = 1, 2, 3$. $\theta \ne \frac{\pi}{12}$ for $i = 6, 7$, 8-1, 9-2. $A = \arccos\left(\frac{1}{4\sin\theta}\right)$, $B = \theta - \arcsin\left(\frac{4\sin^2\theta - 1}{2}\right)$, $C = \frac{\frac{\pi}{2}+\theta+A}{2}$, $C' = \frac{\frac{\pi}{2}+\theta-A}{2}$, $d = \frac{1-4\sin^2\theta}{2}$, $a = \sqrt{4\sin^2\theta - d^2}$, $b = \frac{1-2d}{\tan\theta}$, $c = \frac{\sin^2\theta - \cos^2\theta}{2\sin\theta}$, $m = \frac{d}{\tan\theta} - a$, $m' = \frac{d}{\tan\theta} + a$.



Therefore all the graph representations are given as follows. Here we excluded C because C would give a rhombus but the heavy lines correspond to the longest distance.



Here we consider the embeddings for these graphs. Without loss of generality, we assume the longest distance is 1. Let the vertical angle of the isosceles triangle be 2θ $(0 < \theta < \frac{\pi}{6})$, $A(X) = A_\theta(X) = \{1, \alpha(\theta)(= 2\sin\theta), \beta(\theta)\}$, and the three points of the isosceles triangle be $(0, \cos\theta)$, $(-\sin\theta, 0)$, $(\sin\theta, 0)$. For each of the nine graphs we check the range of θ, the new point $p_i(\theta)$ and new distance $\beta_i(\theta)$. We give this information in Table 1. Note that the last two graphs are embedded in two ways (Fig. 10).

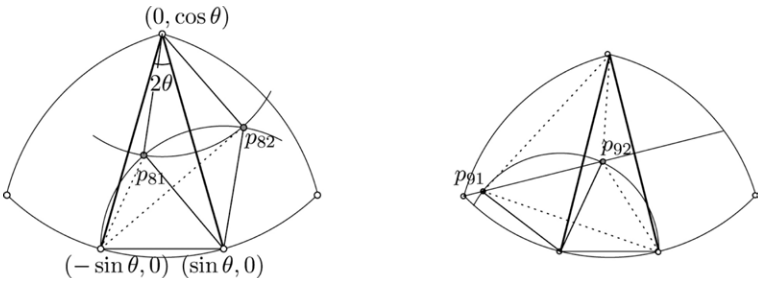


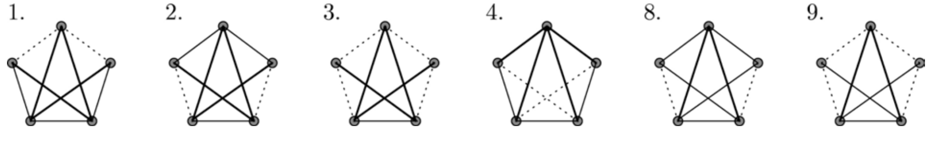
Fig. 10.

Next we find the three-distance set X in question, using the three-distance sets having four points constructed above. If $i \neq 5, 6, 7$, then denote the symmetric point of p_i with respect to the vertical axis by p_i^*. Two points p_i, p_j (resp. p_j^*) can be contained in X if and only if there exists θ such that

$$\begin{cases} \beta_i(\theta) = \beta_j(\theta) \\ d(p_i(\theta), p_j(\theta)) \in A_\theta(X), \ \text{(resp. } d(p_i(\theta), p_j^*(\theta)) \in A_\theta(X)). \end{cases} \tag{1}$$

Assume $i \neq j$, Table 2 shows possible (i, j), θ such that $\beta_i(\theta) = \beta_j(\theta)$, together with the values $\alpha_i(\theta)$, $\beta_i(\theta)$, $d(p_i(\theta), p_j(\theta))$, $d(p_i(\theta), p_j^*(\theta))$. For each such (i, j) we check if it follows the second condition of (1). If it follows this condition, we append a figure number. What has to be noticed is that they contain the three-distance sets which should be excluded in this case, because so far we have used only the assumption that they contain isosceles triangles with two sides having the longest length. We append a symbol * to these three-distance sets. Here we denote the angle θ by degree instead of radian.

Next assume $i = j$ ($i \neq 5, 6, 7$). Now we find the three-distance sets X which do not contain any equilateral triangles and any two-distance set having four points, and such that for each $\gamma \in A(X)$, there exist some isosceles triangles whose two sides have length γ. We check the possibility of the distance between $p_i(\theta)$ and $p_i^*(\theta)$.



For the figs. 1, 3, 4, 8, we cannot obtain the three-distance sets in question. For fig. 2, we only have to check if there exists an angle θ such that $d(p_2(\theta), p_2^*(\theta)) = \beta_2(\theta)$ and for fig. 9, we check if there exists an angle θ such that $d(p_{9k}(\theta), p_{9k}^*(\theta)) = 1 (k = 1, 2)$. When $k = 2$, there exists no such angle θ. For the remaining cases, we have Table 3.

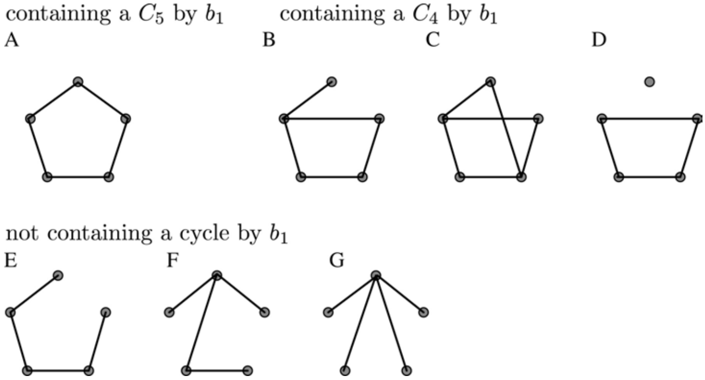
Table 2

$[i, j]$	θ	α	β	$d(i, j)$	$d(i, j^*)$	fig.
[1, 5]	21.502	0.7331	0.3731	0.8608		
[1, 6]	19.108	0.6547	0.5292	0.4932		
[1, 7]	20.705	0.7071	0.4209	0.5952		
[1, 8-2]	15.969	0.5503	0.8350	0.3028	**1**	⋆534
[1, 9-2]	19.504	0.6677	0.5005	0.3342	0.5954	
[2, 4]	12.148	0.4209	0.8229	0.6654	1.164	
[2, 5]	22.860	0.7770	0.3963	0.7008		
[2, 6]	20	0.6840	**0.5321**	**0.5321**		⋆524
[2, 7]	22.500	0.7654	0.4142	0.5858		
[2, 8-2]	15	0.5176	**0.7321**	0.2679	**0.7321**	⋆527
[2, 9-1]	12.857	**0.4450**	0.8019	**1**	**0.4450**	529, 530
[2, 9-2]	20.705	**0.7071**	0.5	0.3536	**0.7071**	531
[3, 4]	10.751	0.3731	0.7331	0.4626	1.240	
[3, 6]	24.677	0.8350	0.5503	0.4595		
[3, 8-2]	14.598	0.5041	0.6661	0.3760	0.7459	
[3, 9-1]	12.148	0.4209	0.7071	**1**	0.2976	⋆511
[4, 6]	7.2992	0.2541	0.5041	0.5645		
[4, 7]	9.7607	0.3391	0.6683	0.7235		
[5, 7]	24.295	0.8229	0.4209	0.1771		
[5, 8-1]	19.408	0.6646	0.3371	0.4556		
[7, 9-2]	12.148	0.4209	0.5952	0.2505		

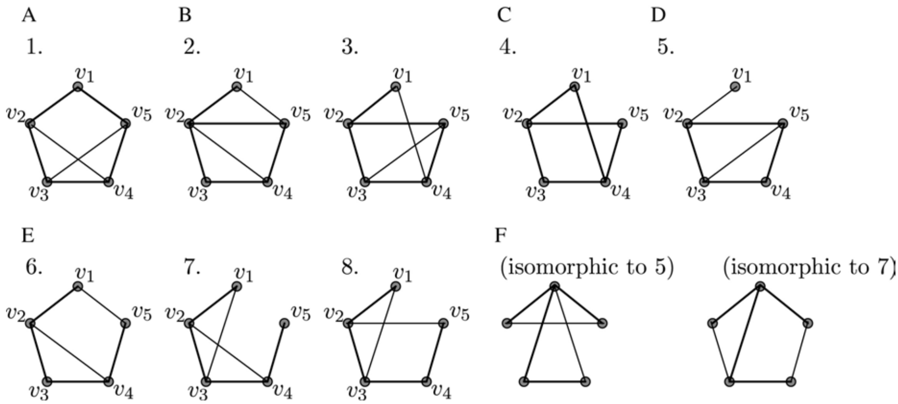
Table 3

i	θ	α	β	d	fig.
2	12.86	0.4450	0.8019	0.8019	Isomorphic to fig. 529
9-1	12.86	0.4450	0.8019	1	532

Secondly, we consider all three colored graphs having K_5 satisfying the conditions (b), that is, those which do not contain one-colored triangles and two colored K_4. Moreover there is a color such that the valency of any vertex with respect to this color is at most one. Let $E_i = \{e \in E(K_n) \mid h(e) = b_i\}$ $(i = 1, 2, 3)$ and $|E_1| \geq |E_2| \geq |E_3|$. We may assume $|E_1| \geq 4$ since $|E(K_5)| = 10 > 3 \cdot 3$. Moreover if $|E_3| \geq 3$, then it does not satisfy the last condition, also if $|E_3| \leq 1$, there exists a two colored K_4 by b_1 and b_2. Hence we have $|E_3| = 2$. First we consider the colorings by b_1. We denote the colorings by heavy lines. These are triangle-free simple graphs of order five and the number of the edges is at least four. All of these graphs are following.



We will prove later, that a graph which contains a subgraph C is not embeddable. Next we consider colorings by b_3, for each graph except C. All of these colorings are the following. Here we denote the coloring by b_3 by thin lines. All the remaining pairs of vertices are joined by b_2 (except the graph C).



For each of the graphs above, we check if it is embedded in $\mathbb{R}^2$.

1. Let $U = \{v_1, v_2, v_3, v_5\}$. Then $\mathrm{Ind}(U)$ is embedded as an isosceles trapezoid or a parallelogram. Assume that the embedding of $\mathrm{Ind}(U)$ is an isosceles trapezoid.

   Since $h(v_3v_4) = h(v_4v_5)$, there is a point p_4 on the perpendicular bisector of p_3p_5 (p_1p_2). Then $d(p_2, p_4) = d(p_1, p_4)$. This is a contradiction. Hence if $\mathrm{Ind}(U)$ is embeddable in $\mathbb{R}^2$, the embedding of $\mathrm{Ind}(U)$ is a parallelogram. Similarly for $U' = \{v_1, v_2, v_4, v_5\}$, the embedding of $\mathrm{Ind}(U')$ is a parallelogram. Consequently $d(p_3, p_4) = 2 \cdot d(p_1, p_3)$ and this embedding is determined uniquely (fig. 533).

2. Let $U = \{v_1, v_2, v_4, v_5\}$. Then $\mathrm{Ind}(U)$ is embedded as an isosceles trapezoid or a parallelogram. Since $h(v_1v_3) = h(v_3v_5)$, $h(v_2v_3) = h(v_3v_4)$, there is a point p_3 on the perpendicular bisector of p_1p_5 and that of p_2p_4.

(i) The case where the embedding of $\mathrm{Ind}(U)$ is an isosceles trapezoid.
Since there is a point p_3 on the perpendicular bisector of p_1p_4, we obtain $d(p_1, p_3) = d(p_3, p_4)$, a contradiction

(ii) The case where the embedding of $\mathrm{Ind}(U)$ is a parallelogram (not a rectangle).

We could not get a point p_3 since the perpendicular bisector of p_1p_5 and that of p_2p_4 do not intersect.
Therefore the graph is not embeddable in $\mathbb{R}^2$.

3. Let $U = \{v_2, v_3, v_4, v_5\}$. $\mathrm{Ind}(U)$ is embedded as a rhombus. Let $d(p_2, p_3) = 1, d(p_2, p_4) = x, d(p_3, p_5) = y$ ($0 < x, y < 2$ and $x, y \neq 1$). Then $x^2 + y^2 = 4$. Since $h(v_1v_3) = h(v_1v_5)$, the three points p_1, p_2, p_4 are collinear.

   (i) The case where the points form a line in order p_1, p_2, p_4.
   In this case
   $$x^2 = \left(1 + \frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2.$$
   So $x = -1, 2$. This is impossible.

   (ii) The case where the points form a line in order p_2, p_1, p_4.
   In this case
   $$x^2 = \left(1 - \frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2.$$
   So $x = 1, -2$. This is impossible.

   (iii) The case where the points form a line in order p_2, p_4, p_1.
   In this case since $x + y = 1$, then $xy = -\frac{3}{2}$. This is impossible.
   Therefore the graph is not embeddable in $\mathbb{R}^2$.

4. Clearly in this case p_1 and p_5 must coincide, which is absurd.
5. Let $U = \{v_2, v_3, v_4, v_5\}$. Since $\mathrm{Ind}(U)$ is embedded as a rhombus, $p_2p_4 \perp p_3p_5$. On the other hand, p_1 is the center of the cycle which passes through the points p_3, p_4 and p_5, since $h(v_1v_3) = h(v_1v_4) = h(v_1v_5)$. Also since $h(v_3v_4) = h(v_4v_5)$, $p_1p_4 \perp p_3p_5$. Then since $h(v_1v_4) = h(v_2v_4)$, $p_1 = p_2$. Therefore the graph is not embeddable in $\mathbb{R}^2$.
6. Let $U = \{v_1, v_2, v_4, v_5\}$. $\mathrm{Ind}(U)$ is embedded as a rectangle. We only have to consider the following two cases, since the coloring for b_1 and b_2 are symmetric.

   (i) The case where diagonals are colored by b_2.
   Let q be the midpoint of p_2p_4, $d(p_1, p_3) = 1$ and $\angle p_1p_3p_5 = 2\theta$. In this case, the range of θ is $0 < \theta < \frac{\pi}{6}$ since p_1p_3 is the longest side. Then $d(p_1, p_5) = d(p_2, p_4) = 2\sin\theta,\ d(p_1, p_2) = \sqrt{1 - 4\sin^2\theta}$. Take notice of the triangle $\triangle p_2p_3q$, then
   $$\left(\cos\theta - \sqrt{1 - 4\sin^2\theta}\right)^2 + \sin^2\theta = 1 - 4\sin^2\theta.$$
   So we have $\cos^2\theta = \frac{3+\sqrt{13}}{8}$. Moreover $\theta \doteqdot 24.6877°$.

(ii) The case where diagonals are colored by b_3.
Since $h(v_2v_3) = h(v_3v_4)$, $h(v_1v_3) = h(v_3v_5)$, the point p_3 is the intersection of the diagonals. Then $d(p_1, p_3) = d(p_2, p_3)$. This is a contradiction. Consequently this embedding is determined uniquely (fig. 534).

7. Let $U = \{v_1, v_2, v_3, v_4\}$. Ind$(U)$ is embedded as an isosceles trapezoid or a parallelogram.

   (i) The case where the embedding of Ind(U) is an isosceles trapezoid.
   Since $h(v_2v_5) = h(v_3v_5)$, there is a point p_5 on the perpendicular bisector of p_2p_3 (p_1p_4). Then $h(v_1v_5) = h(v_4v_5)$, a contradiction.

   (ii) The case where the embedding of Ind(U) is a parallelogram.
   Since $h(v_1v_5) = h(v_2v_5) = h(v_3v_5)$, the point p_5 is the center of the cycle which passes through the points p_1, p_2 and p_3. Hence the radius r equals $d(p_1, p_5) = d(p_1, p_4)$ which is the longest distance. In this case $d(p_4, p_5) > r$ since the point p_4 is outside the cycle. It is impossible. Therefore the graph is not embeddable in $\mathbb{R}^2$.

8. The points p_2 and p_5 are on the perpendicular bisector of p_1p_3. So $p_1p_3 \perp p_2p_5$. Also $h(v_1v_3) = h(v_2v_5)$, $h(v_1v_4) = h(v_2v_4)$ and $h(v_3v_4) = h(v_4v_5)$. Therefore if this graph is embeddable in $\mathbb{R}^2$, then $\triangle p_4p_3p_1$ is obtained by rotating $\triangle p_4p_2p_5$ around p_4 by $\frac{\pi}{2}$ rad. Then $d(p_1, p_4)^2 + d(p_2, p_4)^2 = d(p_1, p_2)^2$ and $d(p_3, p_4)^2 + d(p_4, p_5)^2 = d(p_3, p_5)^2$. It is impossible.
   Therefore the graph is not embeddable in $\mathbb{R}^2$.

Finally we have proved the preceding theorem. We list these three-distance sets in the last section.

4. Three-distance sets of more than five points

In this section we classify three-distance sets having more than five points. Let $\mathbb{X}_i$ be the set of all three-distance sets having i points, $\mathbb{X}_i\{1, \alpha, \beta\}$ be the set of all three-distance sets having i points such that $A(X) = \{1, \alpha, \beta\}$, $A(\mathbb{X}_i)$ be the set of all $A(X)$ for $X \in \mathbb{X}_i$. By Theorem 1, we have known $\mathbb{X}_5$ and $A(\mathbb{X}_5)$. Suppose $X \in \mathbb{X}_i (i \geq 6)$ does not contain the five points of a regular pentagon, then since the regular pentagon is the only two-distance set having five points, we can write $X = X' \cup \{p\}$ where $X' \in \mathbb{X}_5$ so that we have $A(X) = A(X') \in A(\mathbb{X}_5)$. Therefore we only have to classify $\mathbb{X}_i\{1, \alpha, \beta\}(i \geq 6)$ for each $\{1, \alpha, \beta\} \in A(\mathbb{X}_5)$. We have Theorem 2 below.

Theorem 2. (i) *There exists no three-distance set having more than seven points.*

(ii) *There exist only two three-distance sets having seven points (fig.* 701, *fig.* 702*).*

(iii) *There exist only six maximal three-distance sets having six points (figs.* 601–606*).*

(iv) *There exist only sixteen maximal three-distance sets having five points (*†*fig.* 5*).*

In particular, there are exactly twenty four maximal three-distance sets having five or more points.

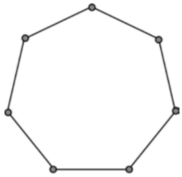


fig. 701

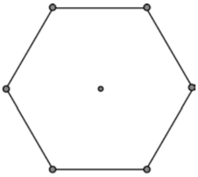
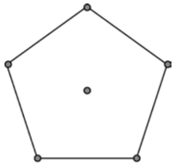


fig. 702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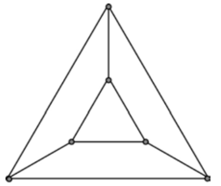
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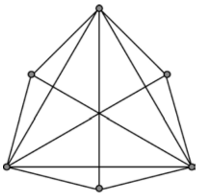
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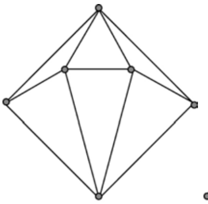
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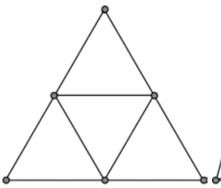
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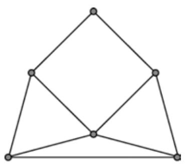
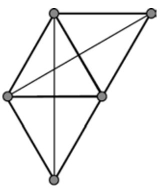


fig. 606

5. Table for three-distance sets having five points

5.1. All three-distance sets having five points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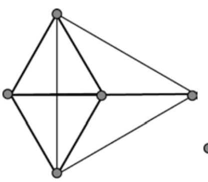
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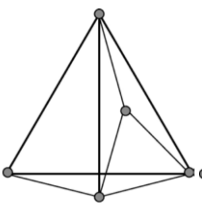
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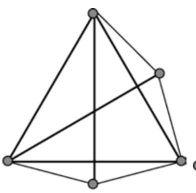
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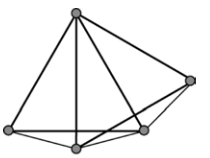
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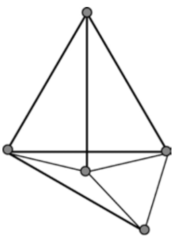
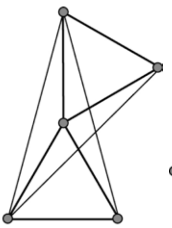

†fig. 506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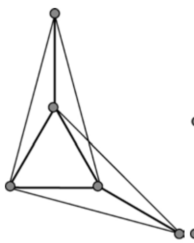
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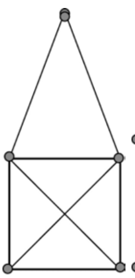
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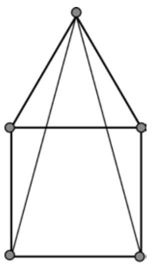
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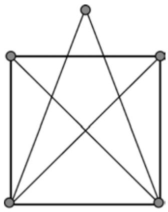
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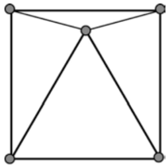
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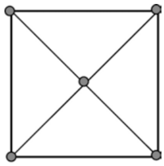
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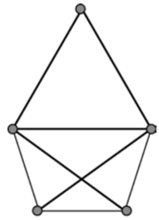
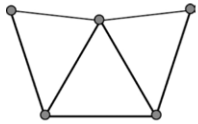
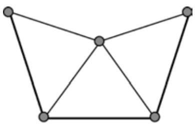

†fig. 514


†fig. 515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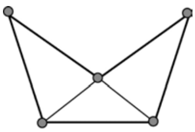
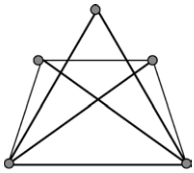

†fig. 517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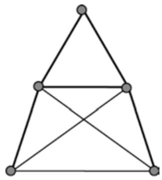
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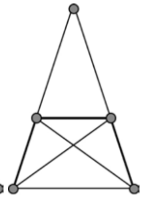
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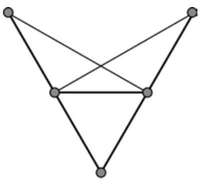
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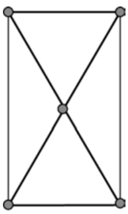
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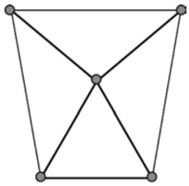
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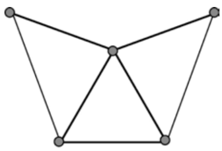
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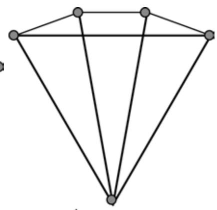
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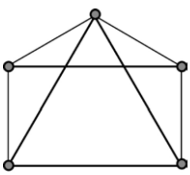
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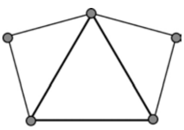
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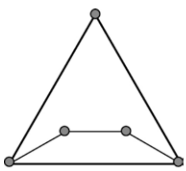
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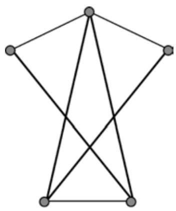
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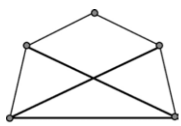
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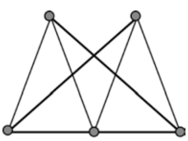
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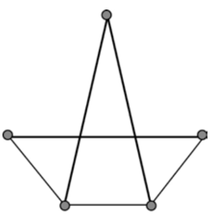
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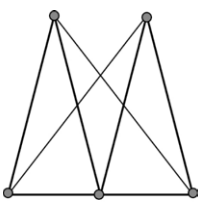
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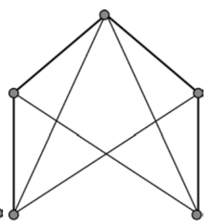
†fig. 534

5.2. Table of $A(X)$

Let $A(X) = \{1, \alpha, \beta\}\,(1 < \alpha < \beta)$. We give α, β in the following table. Here $\gamma = 2\sin\frac{5\pi}{12} = \sqrt{2+\sqrt{3}}$, $\tau = 2\sin\frac{3\pi}{10} = \frac{1+\sqrt{5}}{2}$.

(α, β)	$\|X\| \geq 6$	$\|X\| = 5$
$\left(2\sin\frac{5\pi}{14}, 8\sin^2\frac{5\pi}{14}\sin\frac{3\pi}{14}\right)$	701	529, 530, 532
$(\sqrt{3}, 2)$	702, 605	501, 502, 521, 522, 526
$\left(2\sin\frac{\pi}{5}, 2\tau\sin\frac{\pi}{5}\right)$	601	516
$(\sqrt{2}, \gamma)$	602, 606	503, 504, 507, 510, 527
$(\gamma, \gamma\sqrt{2})$	603, 604	505, 508, 512, 528
(τ, τ^2)		517, 520
$(\sqrt{2}, 2)$		513, 531

(α, β)	fig.	(α, β)	fig.
$(\gamma, 1+\sqrt{3})$	506	$\left(\tau, 2\sin\frac{7\pi}{15}\right)$	519
$(\sqrt{2}, \sqrt{3+\sqrt{7}})$	509	$\left(2\sin\frac{5\pi}{18}, 2\sin\frac{4\pi}{9}\right)$	523
$\left(\frac{\sqrt{6+2\sqrt{7}}}{2}, \sqrt{3+\sqrt{7}}\right)$	511	$\left(2\sin\frac{2\pi}{9}, 2\sin\frac{7\pi}{18}\right)$	524
$\left(\tau, 2\tau\sin\frac{4\pi}{15}\right)$	514	$\left(2\sin\frac{4\pi}{9}, \frac{1}{2\sin\frac{\pi}{18}}\right)$	525
$\left(\frac{2}{\tau}\sin\frac{7\pi}{15}, 2\sin\frac{7\pi}{15}\right)$	515	$(2, \sqrt{6})$	533
$\left(2\sin\frac{4\pi}{15}, 2\tau\sin\frac{4\pi}{15}\right)$	518	$\left(\sqrt{\frac{1+\sqrt{13}}{2}}, \sqrt{\frac{3+\sqrt{13}}{2}}\right)$	534

In the case of two-distance sets, the ratio of the two distances is known to satisfy a very strong condition if the cardinality is sufficiently large [8]. It is an interesting problem to find if a similar result holds for three-distance sets (cf. [1]).

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